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ON A CERTAIN SUBCLASS OF NORMALIZED FUNCTIONS INVOLVING THE RUSAL DIFFERENTIAL OPERATOR

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Abstract

Let N be the class of functions analytic and normalized in open unit disc given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k, a_k \in C$. The objective of this paper is to investigate two subclasses namely $\check{K}(A_\lambda^n; \gamma, \mu, m, \beta)$ and $K(A_\lambda^n; \gamma, \mu, m, \beta)$ operating on subclass of normalized analytical function. RUSAL differential operator are used for finding Closure theorem, Integral mean inequality,extrerme point theorem,coefficient inequality, convolution and distortion theorem for given classes. All the results are sharp for the function mentioned above.

1. Introduction and Preliminaries

Let N denotes subclass of normalized analytical functions in open unit disk $U = \{z : |z| < 1\}$ given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Ruscheweyh in [3] has introduced following differential operator $R^n : N \to N$ defined by

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$$R^n(f(z)) = z + \sum_{k=2}^{\infty} {}^{n+k-1}_{n}C a_k z^k, \quad (z \in U). \tag{1.2}$$

Definition 1.1 : A function f in N is said to be close-to-convex in U, of order α, that is, $f \in C(\alpha)$ if and only if

$$Re\{f(z)\} > \alpha \quad (z \in U). \tag{1.3}$$

Definition 1.2 : A function f in N is said to be close-to-star like of order α $(0 <\leq \alpha < 1)$ that is $f \in CS^*(\alpha)$ if and only if

$$Re\left\{\frac{f(z)}{z}\right\} > \alpha, \quad (z \in U). \tag{1.4}$$

We note that the classes $C(0) = C$, $CS^*(0) = CS^*$ are well known classes of close-to-convex and close-to-star like functions in U.

Definition 1.3 : For two functions f and g analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U and write

$$f(z)g \prec (z), \quad (z \in U) \tag{1.5}$$

If there exit Schwarz function $w(z)$, analytical in U with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)), \quad (z \in U). \tag{1.6}$$

Definition 1.4 : For $f \in N$, [1] has introduced following differential operator D^n : $N \to N$ defined by

$$D^n(f(z)) = z + \sum_{k=2}^{\infty}[1 + (k-1)\partial]^n a_k z^k \quad (z \in U). \tag{1.7}$$

Definition 1.5 : Let $n \in N \cup \{0\}, \lambda \geq 0, A^n_\lambda : N \to N$ defined by

$$A^n_\lambda(f(Z)) = (1-\lambda)D^n f(z) + \lambda R^n f(z). \tag{1.8}$$

On simplifying, we observed that

$$A^n_\lambda(f(z)) = z + \sum_{k=2}^{\infty}([1 + (k-1)\partial]^n(1-\lambda) + \lambda {}^{n+k-1}_{n}C)a_k z^k. \tag{1.9}$$

[6] has introduced] following subordination theorem which we stated as lemma

Lemma 1.1 : Let f and g analytic in unit disc and suppose $g \prec f$, then for $0 < t < \infty$

$$\int_0^{2\pi} |g(re^{i\phi})|^t d\theta \leq \int_0^{2\pi} |f(re^{i\phi})|^t d\theta \quad (0 \leq r < 1, t > 0). \tag{1.11}$$

Strict equality hold for $0 \leq r < 1$ unless f is constant or $w(z) = \alpha z, |\alpha| = 1$.

Definition 1.6 :

$$K(A_\lambda^m; \gamma, \mu, m, \beta), = \left\{ f \in N : \left| \frac{1}{\gamma}((1-u)\frac{A_\lambda^m f}{z} + u(A_\lambda^m f)' - 1) \right| < \beta \right\} \tag{1.12}$$

where $z \in U, \gamma \in C|\{0\}, 0 < \beta \leq 1$ $0 < \mu \leq 1$, $m \in N \cup \{0\}$, $A_\lambda^m f$ is defined in (1.9).

Example : If $f(z) = z$, then for $\gamma = 1, \mu = 1, m = 0, 0 < \beta \leq 1$, show that $f(z) \in K(A_\lambda^m; \gamma, \mu, m, \beta)$.

Answer : For $\gamma = 1, \mu = 1, m = 0, 0 < \nu \leq 1$,

$$\begin{aligned} \left| \frac{1}{\gamma} \left((1-u)\frac{A_\lambda^m f}{z} + u(A_\lambda^m f)' - 1 \right) \right| &= \left| \frac{1}{1} \left((1-1)\frac{A_\lambda^0 f}{z} + u(A_\lambda^0 f)' - 1 \right) \right| \\ &= |(A_\lambda^0 f)' - 1| \\ &= |(z)' - 1| \\ &= |1 - 1| \\ &< \beta. \end{aligned}$$

Hence $f(z) \in K(A_\lambda^m; \gamma, \mu, m, \beta)$.

Definition 1.7 : Let $\check{K}(A_\lambda^n; \gamma, \mu, m, \beta)$ be the subclass of $K(A_\lambda^m; \gamma, \mu, , \beta)$ which satisfies inequality

$$\sum_{k=2}^{\infty} (1 + (k-1)\mu)([1 + (k-1)\partial]^n (1-\lambda) + \lambda^{n+k-1}{}_n C)|a_k| < |\gamma|\beta. \tag{1.13}$$

Remark 1 : $K(A_\lambda^m; 1, 1, 0, \beta) \subseteq C(1-\beta)$.

Remark 2 : $K(A_\lambda^m; 1, 1, 0, \beta) \subseteq CS^*(1-\beta)$.

2. Coefficient Inequality, Growth and Distortion Theorems, Closure Theorems

Theorem 2.1 : Let $f(z) \in N$ satisfy

$$\sum_{k=2}^{\infty} (1 + (k-1)\mu)([1 + (k-1)\partial]^n (1-\lambda) + \lambda^{n+k-1}{}_n C)\ |a_k| < |\gamma|\beta \tag{2.1}$$

$\gamma \in C|\{0\}, 0 < \beta \leq 1 \ \ 0 \leq \mu \leq 1, m \in N \cup \{0\}$.

Then $f \in K(A_\lambda^m; \gamma, \mu, , m, \beta)$.

Proof : Assume (2.1) is valid for $f(z) \in N$ and γ $(\gamma \in C|\{0\}), \beta$ $(0 < \beta \leq)$, μ $(0 < \mu \leq 1)$, $m \in N \cup \{0\}$.

Using (1.8) we have

$$\begin{aligned}
&(1-u)\frac{A_\lambda^m f}{z} + u(A_\lambda^m f)' - 1 \\
&= \frac{(1-u)}{z}\left[z + \sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda) + \lambda^{n+k-1}{}_nC)a_k z^k\right] \\
&+\mu\left[1 + + \sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda) + \lambda^{n+k-1}{}_nC)ka_k z^{k-1}\right] - 1 \\
&= \sum_{k=2}^{\infty}(1+(k-1)\mu)([1+k-1)\partial]^n(1-\lambda) + \lambda^{n+k-1}{}_nC|a_k||z^{-1}|k.
\end{aligned}$$

Therefore

$$\begin{aligned}
&|((1-\mu)\frac{A_\lambda^m f}{z} + \mu(A_\lambda^m f)' - 1)| \\
&= \sum_{k=2}^{\infty}(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda) + \lambda^{n+k-1}{}_nC)|a_k||z^{k-1}| \\
&\leq \sum_{k=2}^{\infty}(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda) + \lambda^{n+k-1}{}_nC)|a_k| \\
&\leq |\gamma|\beta.
\end{aligned}$$

Hence

$$\left|\frac{1}{\gamma}((1-u)\frac{A_\lambda^m f}{z} + u(A_\lambda^m f)' - 1)\right| < \beta.$$

Thus $f(z) \in K(A_\lambda^m; \gamma, \mu, m, \beta)$.

Corollary 2.1 : Show that $\check{K}(A_\lambda^m; \gamma, \mu, m, \beta) \subseteq K(A_\lambda^m; \gamma, \mu, m, \beta)$.

Example : If $f(z) = z + \frac{z^2}{2}$. Show that $f(z) \in K(A_\lambda^m; 1, 1, 0, 1)$ but $f(z) \notin \check{K}(A_\lambda^m; 1, 1, 0, 1)$.

Answer :

$$\begin{aligned}
& \left|\frac{1}{\gamma}((1-u)\frac{A_\lambda^m f}{z}+u(A_\lambda^m f)'-1)\right| \\
= & \left|\frac{1}{1}((1-1)\frac{A_\lambda^0 f}{z}+u(A_\lambda^0 f)'-1)\right| \\
= & \left|(A_\lambda^0 f)'-1\right| \\
= & \left|\left(z+\frac{z^2}{2}\right)'-1\right| \\
= & \left|1+2\frac{z}{2}-1\right| \\
= & |z|<1.
\end{aligned}$$

Therefore $f(z)\in K(A_\lambda^m;1,1,0,1)$. But

$$\begin{aligned}
& \sum_{k=2}^{\infty}(1+(k-1)1)([1+(k-1)\partial]^0(1-\lambda)+\lambda^{0+k-1}{}_nC)|a_k| \\
& =\sum_{k=2}^{\infty}k(1-\lambda+\lambda)|a_k| \\
= & \ 2\cdot\frac{1}{z} \\
= & \ 1\not<1.
\end{aligned}$$

Therefore $f(z)\notin \check{K}(A_\lambda^m;1,1,0,1)$.

Theorem 2.2 : If $f(z)\in \check{K}(A_\lambda^m;\gamma,\mu,m,\beta)$ then

$$|a_k|\le\frac{|\gamma|\beta}{(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)}\quad k\ge 2.$$

Proof : given that $f(z)\in \check{K}(A_\lambda^m;\gamma,\mu,m,\beta)$. Therefore

$$\begin{aligned}
& \sum_{k=2}^{\infty}(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)|a_k|\le|\gamma|\beta \\
& (1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)|a_k|\le|\gamma|\beta \\
& |a_k|\le\frac{|\gamma|\beta}{(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)}.
\end{aligned}$$

Theorem 2.3 : Let function $f(z)$ defined by (1.1) be in class $\check{K}(A_\lambda^m;\gamma,\mu,m,\beta)$, then

$$|z|-\frac{|\gamma|\beta}{[1+\mu]((1+\partial)^n(1-\lambda)+\lambda^{n+1}{}_nC)}|z|^2\le|f(z)|$$

$$\leq |z| + \frac{|\gamma|\beta}{[1+\mu]((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ n}C)}|z|^2. \tag{2.2}$$

Equality is attained for function $f(z)$ given by

$$f(z) = z + \frac{|\gamma|\beta}{[1+\mu]((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ n}C)}z^2.$$

Proof :

$$\begin{aligned}
&[[(1+\mu)(1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ n}C)\sum_{k=2}^{\infty}|a_k| \\
&\leq \sum_{k=2}^{\infty}(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ n}C(|a_k| \\
&\leq |||\beta.
\end{aligned}$$

Therefore

$$\sum_{k=2}^{\infty}|a_k| \leq \frac{|\gamma|\beta}{[1+\mu]((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ n}C)}. \tag{2.3}$$

Alsom $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$ and using (2.3)

$$\begin{aligned}
|f(z)| &\leq |z| + \sum_{k=2}^{\infty}|a_k||z^k| \\
&\leq |z| + |z|^2\sum_{k=2}^{\infty}|a_k| \qquad (2.4) \\
&\leq |z| + |z|^2\tfrac{|\gamma|\beta}{[1+\mu]((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ n}C)}.
\end{aligned}$$

Similarly

$$\begin{aligned}
|f(z)| &\geq |z| - \sum_{k=2}^{\infty}|a_k||z^k| \\
&\geq |z| - |z|^2\sum_{k=2}^{\infty}|a_k| \qquad (2.5) \\
&\geq |z| - |z|^2\tfrac{|\gamma|\beta}{[1+\mu]((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ n}C)}.
\end{aligned}$$

Using (2.4) and (2.5)

$$|z| - \frac{|\gamma|\beta}{[1+\mu]((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ n}C)}|z|^2 \leq |f(z)|$$

$$\leq |z| + \frac{|\gamma|\beta}{[1+\mu]((1+\partial)^n(1-\lambda)+\lambda^n{}_n^{+1}C)}|z|^2.$$

Hence

$$|z| - \frac{|\gamma|\beta}{[1+\mu]((1+\partial)^n(1-\lambda)+\lambda^n{}_n^{+1}C)}|z|^2$$

$$\leq |f(z)| \leq |z| + \frac{|\gamma|\beta}{[1+\mu]((1+\partial)^n(1-\lambda)+\lambda^n{}_n^{+1}C)}|z|^2.$$

Equality is attained for fucntion f(z) given by

$$f(z) = z + \frac{|\gamma|\beta}{[1+\mu]((1+\partial)^n(1-\lambda)+\lambda^n{}_n^{+1}C)}z^2.$$

Theorem 2.4 : Let function $f(z)$ defined by (1.1) be in class $\check{K}(A_\lambda^m;\gamma,\mu,m,\beta)$, then

$$1 - \frac{2|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda^n{}_n^{+1}C)}|z| \leq |f'(z)| \leq 1 + \frac{2|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda^n{}_n^{+1}C)}|z|. \tag{2.6}$$

Equality attained for function $f(z)$ given by

$$f(z) = z + \frac{|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda^n{}_n^{+1}C)}z^2.$$

Proof : Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $f'(z) = 1 + \sum_{k=2}^{\infty} k a_k z^{k-1}$

$$|f'(z)| \leq 1 + \sum_{k=2}^{\infty} k|a_k||z^{k-1}| \leq 1 + |z|\sum_{k=2}^{\infty} k|a_k|. \tag{2.7}$$

But

$$\sum_{k=2}^{\infty}(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)|a_k| \leq |\gamma|\beta.$$

Also

$$\begin{aligned}
&2+(k-2)\mu \geq 0\\
&2+k\mu-2\mu \geq 0\\
&2+2k\mu-2\mu \geq k\mu\\
&\frac{k\mu}{2} \leq 1+(k-1)\mu.
\end{aligned}$$

Similarly

$$
\begin{aligned}
&(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ \ \ n}C(|a_k| \\
&\geq \frac{k\mu}{2}((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C) \\
&\sum_{k=2}^{\infty}\frac{k\mu}{2}((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)|a_k| \\
&\leq \sum_{k=2}^{\infty}(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ \ \ n}C)|a_k| \\
&\leq |\gamma|\beta.
\end{aligned}
$$

$$\sum_{k=2}^{\infty} k|a_k| \leq \frac{2|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)}.$$

From (2.7)

$$|f'(z)| \leq 1+\frac{2|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)}|z|. \tag{2.8}$$

Similarly,

$$
\begin{aligned}
|f'(z)| &\geq 1-\sum_{k=2}^{\infty}k|a_k||z^{k-1}| \\
&\geq 1-|z|\sum_{k=2}^{\infty}k|a_k| \\
&\geq 1-\frac{2|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)}|z|
\end{aligned}
\tag{2.9}
$$

(2.8) and (2.9) implies that

$$1-\frac{2|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)}|z| \leq |f'(z)| \leq 1+\frac{2|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)}|z|.$$

Equality attained for function $f(z)$ given by

$$f(z) = z+\frac{2|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)}z^2.$$

3. Closure Theorem

Theorem 3.1 : Let $f_j() = z+\sum_{k=2}^{\infty}a_{k,j}z^k$, $f_j(z)\in \check{K}(A^m_\lambda;\gamma,\mu,m,\beta)$, then for $g(z) = \sum_{k=1}^{l}c_j\ f_j(z)$.

$$g(z)\in \check{K}(A^m_\lambda;\gamma,\mu,m,\beta), \quad \text{where} \quad \sum_{j=1}^{l}c_j = 1.$$

Proof : Let $f_j(z) = z + \sum\limits_{k=2}^{\infty} a_{k,j} z^k$, with $f_j(z) \in \check{K}(A_\lambda^m; \gamma, \mu, m, \beta)$.

$$\sum_{k=2}^{\infty}[1+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)|a_{k,j}| \leq |\gamma|\beta.$$

$$\begin{aligned}
g(z) &= \sum_{k=1}^{l} c_j \ f_j(z) \\
&= \sum_{k=1}^{l} c_j \left(z + \sum_{k=2}^{\infty} a_{k,j} z^k \right) \\
&= z + \sum_{k=1}^{l} c_j \left(z + \sum_{k=2}^{\infty} a_{k,j} z^k \right) \\
&= z + \sum_{k=2}^{\infty} z^k \sum_{j=1}^{l} c_j a_{k,j} \\
&= z + \sum_{k=2}^{\infty} e_k z^k \quad \text{where} \quad e_k = \sum_{j=1}^{l} c_j a_{k,j}.
\end{aligned}$$

Claim : $g(z) \in \check{K}(A_\lambda^m; \gamma, \mu, m, \beta)$.

$$\begin{aligned}
&\sum_{k=2}^{\infty}[1+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)|e_k| \\
&\sum_{k=2}^{\infty}[1+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC) \left| \sum_{j=1}^{l} c_j a_{k,j} \right| \\
&\leq \sum_{j=1}^{l} \left(c_j \sum_{k=2}^{\infty}[1+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)|a_{k,j}| \right) \\
&\leq \sum_{j=1}^{l} c_j |\gamma|\beta \\
&\leq |\gamma|\beta.
\end{aligned}$$

Therefore, $g(z) \in \check{K}(A_\lambda^m; \gamma, \mu, m, \beta)$.

4. Extreme Point Theorem

Remark 4.1 : For $\gamma \in C|\{0\}$, $0 < \beta \leq 1, 0 \leq \mu \leq 1, m \in N \cup \{0\}$ the following functions are in Class $\check{K}(A^m_\lambda; \gamma, \mu, m, \beta)$.

$$\begin{aligned} f_1(z) &= z + \frac{\beta|\gamma|}{(1+\mu)((1+\partial)^n(1-\lambda)+\lambda^{n+1}{}_nC)}\ z^2, \quad (z \in U) \\ f_2(z) &= z + \frac{2(|\gamma|\beta-1)}{(1+2\mu)((1+2\partial)^n(1-\lambda)+\lambda^{n+2}{}_nC)}\ z^3, \quad (z \in U) \\ f_3(z) &= z + \frac{z^2}{(1+\mu)((1+\partial)^n(1-\lambda)+\lambda^{n+1}{}_nC)} \\ &\quad + \frac{(|\gamma|\beta-1)}{(1+2\mu)((1+2\partial)^n(1-\lambda)+\lambda^{n+2}{}_nC)}\ z^2, \quad (z \in U). \end{aligned}$$

Theorem 4.1 : Let $f_1(z) = z$ and

$$f_k(z) = z + \frac{|\gamma|\beta}{(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)}\ z^k, \quad (k \geq 2). \quad (4.1)$$

Then $f(z) \in \check{K}(A^m_\lambda; \gamma, \mu, m, \beta)$ if and only if $(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$ whee $\lambda_k \geq 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof : Suppose that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} \lambda_k f_k(z) \\ &= \lambda_1 f_1(z) + \sum_{k=2}^{\infty} \lambda_k \left(z + \frac{|\gamma|\beta}{(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)} z^2 \right) \\ &= \left(1 - \sum_{k=2}^{\infty} \lambda_k\right) z + \sum_{k=2}^{\infty} \lambda_k \left(z + \frac{|\gamma|\beta}{(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)} z^k \right) \\ &= z + \sum_{k=2}^{\infty} \lambda_k z^k \frac{|\gamma|\beta}{(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)} \\ &= z + \sum_{k=2}^{\infty} a_k z^k \end{aligned}$$

where

$$a_k = \frac{|\gamma|\beta}{(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)} \lambda_k.$$

Claim : $f(z) \in \check{K}(A_\lambda^m; \gamma, \mu, m, \beta)$.

$$\begin{aligned}
&\sum_{k=2}^{\infty}[1+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)|a_k| \\
&\sum_{k=2}^{\infty}[1+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC) \\
&\frac{|\gamma|\beta}{(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)}\lambda_k \\
&= |\gamma|\beta\sum_{k=2}^{\infty}\lambda_k \\
&= |\gamma|\beta(1-\lambda_1) \\
&\leq |\gamma|\beta.
\end{aligned}$$

From Theorem 2.1 $f(z) \in \check{K}(A_\lambda^m; \gamma, \mu, m, \beta)$.

Conversely suppose that $f(z) \in \check{K}(A_\lambda^m; \gamma, \mu, m, \beta)$. Setting

$$\lambda_k = \frac{[1+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)}{|\gamma|\beta}a_k$$

and $\lambda_1 = \sum_{k=2}^{\infty}\lambda_k$

$$\begin{aligned}
&\sum_{k=1}^{\infty}\lambda_k f_k(z) = \lambda_1 f_1(z) + \sum_{k=2}^{\infty}\lambda_k f_k(z) \\
= \quad & \left(1-\sum_{k=2}^{\infty}\lambda_k\right)z + \sum_{k=2}^{\infty}\lambda_k\left(z + \frac{|\gamma|\beta}{(1+(k-1)\mu)([1+(K-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)}z^k\right) \\
= \quad & z + \sum_{k=2}^{\infty}\lambda_k z^k\frac{|\gamma|\beta}{(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)}z^k \\
= \quad & z + \sum_{k=2}^{\infty}a_k z^k\frac{(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)a^k}{|\gamma|\beta} \\
& \frac{|\gamma|\beta}{(1+(k-1)\mu)([1+(K-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)}z^k \\
= \quad & z + \sum_{k=2}^{\infty}a_k z^k \\
= \quad & f(z).
\end{aligned}$$

Hence $f(z) = \sum\limits_{k=1}^{\infty}\lambda_k f_k(z)$.

5. Integral Mean Inequality for Differential Operator

Theorem 5.1 : $f(z) \in \check{K}(A_\lambda^m; \gamma, \mu, m, \beta)$ and suppose that

$$\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)|a_k| \leq \frac{|\gamma|\beta}{(1+(j-1)\mu)}. \tag{5.1}$$

Also let the function

$$f_j(z) = z + \frac{|\gamma|\beta}{(1+(j-1)\mu)([1+(j-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)} z^j \quad (j \geq 2). \tag{5.2}$$

If ther exists an analytic function $w(z)$ given by

$$w(z)^{j-1} = \frac{(1+(j-1)\mu)}{|\gamma|\beta}\sum_{k=2}^{\infty}((1-la)[1+(k-1)\partial]^n+\lambda^{n+k-1}{}_nC)a_k z^{k-1}. \tag{5.3}$$

Then for $z = re^{i\theta}$ with $0 < r < 1$

$$\int_0^{2\pi} |A_\lambda^n f(z)|^t d\theta \leq \int_0^{2\pi} |A_\lambda^n f_j(z)|^t d\theta \quad (0 \leq \lambda \leq 1,\ t > 0)$$

where $A_\lambda^n f$ is differential operator defined in (1.5).

Proof : We have from definition (1.5)

$$\begin{aligned} A_\lambda^n f(z) &= z + \sum_{k=2}^{\infty}((1-\lambda)[1+(k-1)\partial]^n \lambda^{n+k-1}{}_nC)a_k z^k \\ D_\lambda^n f_j(z) &= z + \frac{|\gamma|\beta}{(1+(j-1)\mu)} z^j. \end{aligned}$$

For $z = re^{i\theta}$ with $0 < r < 1$ we have to show that

$$\begin{aligned} &\int_0^{2\pi} \left| 1 + \sum_{k=2}^{\infty}[(1-\lambda)[1+(k-1)\partial]^n \lambda^{n+k-1}{}_nC]a_k z^{k-1} \right|^t d\theta \\ &\leq \int_0^{2\pi} \left| 1 + \frac{|\gamma|\beta}{(1+(j-1)\mu)} z^{j-1} \right|^t d\theta \qquad (t > 0). \end{aligned}$$

By applying Littlewoods subordination theorem, it would sufficient to show that

$$1 + \sum_{k=2}^{\infty}[(1-\lambda)][1+(k-1)\partial]^n+\lambda^{n+k-1}{}_nC)|a_k|z^{k-1} \prec 1 + \frac{|\gamma|\beta}{(1+(j-1)\mu)} z^{j-1}.$$

That is $t(z) \prec h(z)$ where

$$t(z) = 1 + \sum_{k=2}^{\infty}[(1-\lambda)[1+(k-1)\partial]^n+\lambda^{n+k-1}{}_nC]a_k z^{k-1}$$

$$h(z) = 1 + \frac{|\gamma|\beta}{(1+(j-1)\mu)} z^{j-1}.$$

That is we want to show that $t(z) = h(w(z))$, $w(0) = 0$ and $|w(z) \leq 1$.

$$\begin{aligned} h(w(z)) &= 1 + \frac{|\gamma|\beta}{(1+(j-1)\mu)} w(z)^{j-1}. \\ &= 1 + \frac{|\gamma|\beta}{(1+(j-1)\mu)} \frac{(1+(j-1)\mu)}{|\gamma|\beta} \\ &\quad \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda^{n+k-1}{}_nC] a_k z^{k-1} \\ &= 1 + \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda^{n+k-1}{}_nC] a_k z^{k-1} \\ &= t(z). \end{aligned}$$

Therefore $h(w(z)) = t(z)$ and $w(0) = 0$.

Moreover we prove that analytic function $|w(z)| < 1, z \in U$

$$\begin{aligned} &\left|w(z)^{j-1}\right| = \left| \frac{(1+(j-1)\mu)}{|\gamma|\beta} \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda^{n+k-1}{}_nC] a_k z^{k-1} \right| \\ &\leq \frac{(1+(j-1)\mu)}{|\gamma|\beta} \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda^{n+k-1}{}_nC] |a_k| |z^{k-1}| \\ &\leq |z| \frac{(1+(j-1)\mu)}{|\gamma|\beta} \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda^{n+k-1}{}_nC] |a_k| \\ &\leq |z| < 1 \quad \text{by hypothesis (5.1).} \end{aligned}$$

Hence proved.

6. Convolution Theorems

Definition 6.1 : If $f(z) = z + \sum_{n=2}^{\infty} a_k z^k$, $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, then hadmad product is defined as given below

$$f * g = z + \sum_{k=2}^{\infty} (a_k b_k) z^k.$$

Theorem 6.1 : Let $f, g \in \check{K}(A_\lambda^m; \gamma, \mu, m, \beta)$ where $f(z) = z + \sum_{n=2}^{\infty} a_k z^k$, $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ with $a_k \geq 0$, $b_k \geq 0$ and $(a_k b_k)^{\frac{1}{2}} < 1$. Then $f * g \in \check{K}(A_\lambda^m; \gamma, \mu, m, \beta)$.

Proof : We have $f * g \in \check{K}(A^m_\lambda;\gamma,\mu,m,\beta)$ if and only if

$$\sum_{k=2}^{\infty}[1+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ \ \ n}C)|a_k| \leq |\gamma|\beta$$

$g \in\in \check{K}(A^m_\lambda;\gamma,\mu,m,\beta)$ if and only if

$$\sum_{k=2}^{\infty}[1+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ \ \ n}C)|b_k| \leq |\gamma|\beta.$$

By Cauchy Schwarz inequality

$$\sum_{k=2}^{\infty}(t_k|a_k||t_k|b_k)^{\frac{1}{2}} \leq \left(\sum_{k=2}^{\infty}t_k|a_k|\right)^{\frac{1}{2}}\sum_{k=2}^{\infty}t_k|b_k|)^{\frac{1}{2}}$$

where

$$t_k=(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ \ \ n}C)$$

$$\sum_{k=2}^{\infty}(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ \ \ n}C)|a_kb_k|^{\frac{1}{2}} \leq |\gamma|\beta. \tag{6.2}$$

By assumption $(a_kb_k)^{\frac{1}{2}} < 1.$ then

$$a_kb_k < (a_kb_k)^{\frac{1}{2}}.$$

Thus from (6.2) and(6.3)

$$\sum_{k=2}^{\infty}(1+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ \ \ n}C)|a_kb_k| < |\gamma|\beta.$$

Hence $f * g \in \check{K}(A^m_\lambda;\gamma,\mu,m,\beta)$.

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