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ZERO-FREE REGIONS FOR POLAR DERIVATIVE OF POLYNOMIALS WITH RESTRICTED COEFFICIENTS

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Abstract

In this paper we prove some interesting generalizations of the Enestrm-Kakeya theorem and obtain zero-free regions for polar derivatives of polynomials with restricted coefficients.

1. Introduction

The well known result the Eneström-Kakeya theorem [3,5] in the theory of distribution of zeros of polynomials is the following.

Theorem A : Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0 < a_0 \leq a_1 \leq \ldots \leq a_n$ then all the zeros of $P(z)$ lie in $|z| \leq 1$.

Applying the above result to the polynomial $z^n P(\frac{1}{z})$, we get the following result:

Key Words : Zeros of polynomial, Eneström-Kakeya theorem, Polar derivative.

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If $P(z)=\sum\limits_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0<a_n \leq a_{n-1} \leq \cdots \leq a_2 \leq a_1 \leq a_0$ then $P(z)$ does not vanish in $|z|<1$.

In the literature [1-2, 4 and 6-7] there are several extensions and generalizations of the Enestrom-Kakeya Theorem. The polar derivative of $P(z)$ with respect to real number α is given by $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$.

In this paper we prove the following results:

Theorem 1 : Let $P(z)=\sum\limits_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 3$ with real coefficients and let $D_\alpha P(z)$ be the polar derivative of $P(z)$ with respect to a real number α such that for $n\alpha a_n + a_{n-1} \neq 0$,

$$\begin{aligned} n\alpha a_n + a_{n-1} &\geq (n-1)\alpha a_{n-1} + 2a_{n-2} \leq (n-2)\alpha a_{n-2} + 3a_{n-3} \geq ... \leq 4\alpha a_4 + (n-3)a_3 \\ &\geq 3\alpha a_3 + (n-2)a_2 \leq 2\alpha a_2 + (n-1)a_1 \geq \alpha a_1 + na_0 \ if\ n\ is\ even,\ (OR) \\ n\alpha a_n + a_{n-1} &\geq (n-1)\alpha a_{n-1} + 2a_{n-2} \leq (n-2)\alpha a_{n-2} + 3a_{n-3} \geq ... \geq 4\alpha a_4 + (n-3)a_3 \\ &\leq 3\alpha a_3 + (n-2)a_2 \geq 2\alpha a_2 + (n-1)a_1 \leq \alpha a_1 + na_0 \ if\ n\ is\ odd, \end{aligned}$$

then $D_\alpha P(z)$ does not vanish in the disk

$$(i)\ |z| < \frac{1}{|na_0+\alpha a_1|}\left[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + X_1 - (na_0+\alpha a_1)\right] \ if\ n\ is\ even,$$

$$\begin{aligned} where\ X_1 = 2\Big[&\big\{[(n-2)\alpha a_{n-2}+3a_{n-3}] + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\big\} \\ &- \big\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + [(n-3)\alpha a_{n-3} + 4a_{n-4}] + ... + [3\alpha a_3 + (n-2)a_2]\big\}\Big]. \end{aligned}$$

$$(ii)\ |z| < \frac{1}{|na_0+\alpha a_1|}\left[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1} + X_2 + (na_0+\alpha a_1)\right] \ if\ n\ is\ odd,$$

$$\begin{aligned} where\ X_2 = 2\Big[&\big\{[(n-2)\alpha a_{n-2}+3a_{n-3}] + ... + [5\alpha a_5 + (n-4)a_4] + [3\alpha a_3 + (n-2)a_2]\big\} \\ &- \big\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + [(n-3)\alpha a_{n-3} + 4a_{n-4}] \\ &+ ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\big\}\Big]. \end{aligned}$$

Corollary 1 : Let $P(z)=\sum\limits_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 3$ with real coefficients and let $D_0P(z)$ be the polar derivative of $P(z)$ such that for $a_{n-1} \neq 0$,

$a_{n-1} \geq 2a_{n-2} \leq 3a_{n-3} \geq ... \leq (n-3)a_3 \geq (n-2)a_2 \leq (n-1)a_1 \geq na_0$ if n is even, (OR)

$a_{n-1} \geq 2a_{n-2} \leq 3a_{n-3} \geq ... \geq (n-3)a_3 \leq (n-2)a_2 \geq (n-1)a_1 \leq na_0$ if n is odd, *then $D_0P(z)$ does not vanish in the disk*

(i) $|z| < \dfrac{1}{|na_0|}\left[|a_{n-1}| + a_{n-1} + X_3 - na_0\right]$ if n is even,

where $X_3 = 2\left[\left\{3a_{n-3} + ... + (n-3)a_3 + (n-1)a_1\right\} - \left\{2a_{n-2} + 4a_{n-4} + ... + (n-2)a_2\right\}\right]$.

$(ii)|z| < \dfrac{1}{|na_0|}\left[|a_{n-1}| + a_{n-1} + X_4 + na_0\right]$ if n is odd, where

$X_4 = 2\left[\left\{3a_{n-3} + ... + (n-4)a_4 + (n-2)a_2\right\} - \left\{2a_{n-2} + 4a_{n-4} + ... + (n-3)a_3 + (n-1)a_1\right\}\right]$.

Corollary 2 : Let $P(z) = \sum\limits_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 3$ with real coefficients and let $D_\alpha P(z)$ be the polar derivative of $P(z)$ with respect to a real number α such that for $n\alpha a_n + a_{n-1} \neq 0$,

$$\begin{aligned} n\alpha a_n + a_{n-1} &\geq (n-1)\alpha a_{n-1} + 2a_{n-2} \leq (n-2)\alpha a_{n-2} + 3a_{n-3} \geq ... \leq 4\alpha a_4 + (n-3)a_3 \\ &\geq 3\alpha a_3 + (n-2)a_2 \leq 2\alpha a_2 + (n-1)a_1 \geq \alpha a_1 + na_0 \; if \; n \; is \; even, \; (OR) \\ n\alpha a_n + a_{n-1} &\geq (n-1)\alpha a_{n-1} + 2a_{n-2} \leq (n-2)\alpha a_{n-2} + 3a_{n-3} \geq ... \geq 4\alpha a_4 + (n-3)a_3 \\ &\leq 3\alpha a_3 + (n-2)a_2 \geq 2\alpha a_2 + (n-1)a_1 \leq \alpha a_1 + na_0 \; if \; n \; is \; odd, \end{aligned}$$

then $D_\alpha P(z)$ does not vanish in the disk

$$(i)\ |z| < \frac{1}{na_0 + \alpha a_1}\left[2(n\alpha a_n + a_{n-1}) + X_1 - (na_0 + \alpha a_1)\right] \; if \; n \; is \; even,$$

$$\begin{aligned} where\ X_1 = 2\Big[&\left\{[(n-2)\alpha a_{n-2} + 3a_{n-3}] + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\right\} \\ &- \left\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + [(n-3)\alpha a_{n-3} + 4a_{n-4}] + ... + [3\alpha a_3 + (n-2)a_2]\right\}\Big]. \end{aligned}$$

$$(ii)\ |z| < \frac{1}{na_0 + \alpha a_1}\left[2(n\alpha a_n + a_{n-1} + X_2 + (na_0 + \alpha a_1)\right] \; if \; n \; is \; odd,$$

$$\begin{aligned} where\ X_2 = 2\Big[&\left\{[(n-2)\alpha a_{n-2} + 3a_{n-3}] + ... + [5\alpha a_5 + (n-4)a_4] + [3\alpha a_3 + (n-2)a_2]\right\} \\ &- \left\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + [(n-3)\alpha a_{n-3} + 4a_{n-4}] \right. \\ &\left. + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\right\}\Big]. \end{aligned}$$

Remark 1 : By taking $\alpha = 0$ in Theorem 1, it reduces to Corollary 1.

Remark 2 : By taking $(i+1)\alpha a_{i+1} + (n-i)a_i > 0$ for $i = 0, 1, 2, ..., n-2$, in Theorem 1, it reduces to Corollary 2.

Theorem 2 : Let $P(z)=\sum\limits_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 3$ with real coefficients and let $D_\alpha P(z)$ be the polar derivative of $P(z)$ with respect to a real number α such that for $n\alpha a_n + a_{n-1} \neq 0$,

$$n\alpha a_n + a_{n-1} \leq (n-1)\alpha a_{n-1} + 2a_{n-2} \geq (n-2)\alpha a_{n-2} + 3a_{n-3} \leq ... \geq 4\alpha a_4 + (n-3)a_3$$
$$\leq 3\alpha a_3 + (n-2)a_2 \geq 2\alpha a_2 + (n-1)a_1 \leq \alpha a_1 + na_0 \text{ if n is even, (OR)}$$
$$n\alpha a_n + a_{n-1} \leq (n-1)\alpha a_{n-1} + 2a_{n-2} \geq (n-2)\alpha a_{n-2} + 3a_{n-3} \leq ... \leq 4\alpha a_4 + (n-3)a_3$$
$$\geq 3\alpha a_3 + (n-2)a_2 \leq 2\alpha a_2 + (n-1)a_1 \geq \alpha a_1 + na_0 \text{ if n is odd,}$$

then $D_\alpha P(z)$ does not vanish in the disk
(i) $|z| < \frac{1}{|na_0+\alpha a_1|}\big[|n\alpha a_n + a_{n-1}| - (n\alpha a_n + a_{n-1}) - X_1 + (na_0 + \alpha a_1)\big]$ if n is even, where

$$X_1 = 2\Big[\big\{[(n-2)\alpha a_{n-2} + 3a_{n-3}] + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\big\}$$
$$-\big\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + [(n-3)\alpha a_{n-3} + 4a_{n-4}] + ... + [3\alpha a_3 + (n-2)a_2]\big\}\Big].$$

(ii) $|z| < \frac{1}{|na_0+\alpha a_1|}\big[|n\alpha a_n + a_{n-1}| - (n\alpha a_n + a_{n-1}) - X_2 - (na_0 + \alpha a_1)\big]$ if n is odd, where

$$X_2 = 2\big[\big\{[(n-2)\alpha a_{n-2} + 3a_{n-3}] + ... + [5\alpha a_5 + (n-4)a_4] + [3\alpha a_3 + (n-2)a_2]\big\}$$
$$-\big\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + [(n-3)\alpha a_{n-3} + 4a_{n-4}] + ... + [4\alpha a_4 + (n-3)a_3]$$
$$+[2\alpha a_2 + (n-1)a_1]\big\}\big].$$

Corollary 3 : Let $P(z)=\sum\limits_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 3$ with real coefficients and let $D_0P(z)$ be the polar derivative of $P(z)$ such that for $a_{n-1} \neq 0$,
$a_{n-1} \leq 2a_{n-2} \geq 3a_{n-3} \leq .... \geq (n-3)a_3 \leq (n-2)a_2 \geq (n-1)a_1 \leq na_0$ if n is even, (OR)
$a_{n-1} \geq 2a_{n-2} \leq 3a_{n-3} \geq ... \leq (n-3)a_3 \geq (n-2)a_2 \leq (n-1)a_1 \geq na_0$ if n is odd,
then $D_0P(z)$ does not vanish in the disk.
(i) $|z| < \frac{1}{|na_0|}\big[|a_{n-1}| - a_{n-1} - X_3 + na_0\big]$ if n is even, where

$$X_3 = 2\big[\big\{3a_{n-3} + ... + (n-3)a_3 + (n-1)a_1\big\} - \big\{2a_{n-2} + 4a_{n-4} + ... + (n-2)a_2\big\}\big].$$

(ii) $|z| < \frac{1}{|na_0|}\big[|a_{n-1}| - a_{n-1} - X_4 - na_0\big]$ if n is odd, where

$$X_4 = 2\big[\big\{3a_{n-3} + ... + (n-4)a_4 + (n-2)a_2\big\} - \big\{2a_{n-2} + 4a_{n-4} + ... + (n-3)a_3 + (n-1)a_1\big\}\big].$$

Corollary 4 : Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 3$ with real coefficients and let $D_\alpha P(z)$ be the polar derivative of $P(z)$ with respect to a real number α such that for $n\alpha a_n + a_{n-1} \neq 0$,

$$n\alpha a_n + a_{n-1} \leq (n-1)\alpha a_{n-1} + 2a_{n-2} \geq (n-2)\alpha a_{n-2} + 3a_{n-3} \leq ... \geq 4\alpha a_4 + (n-3)a_3$$
$$\leq 3\alpha a_3 + (n-2)a_2 \geq 2\alpha a_2 + (n-1)a_1 \leq \alpha a_1 + na_0 \text{ if n is even, (OR)}$$
$$n\alpha a_n + a_{n-1} \leq (n-1)\alpha a_{n-1} + 2a_{n-2} \geq (n-2)\alpha a_{n-2} + 3a_{n-3} \leq ... \leq 4\alpha a_4 + (n-3)a_3$$
$$\geq 3\alpha a_3 + (n-2)a_2 \leq 2\alpha a_2 + (n-1)a_1 \geq \alpha a_1 + na_0 \text{ if n is odd,}$$

then $D_\alpha P(z)$ does not vanish in the disk
(i) $|z| < \frac{1}{na_0+\alpha a_1}\big[(na_0 + \alpha a_1) - X_1\big]$ if n is even, where

$$\begin{aligned} X_1 \;&=\; 2\Big[\big\{[(n-2)\alpha a_{n-2} + 3a_{n-3}] + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\big\} \\ &\quad -\big\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + [(n-3)\alpha a_{n-3} + 4a_{n-4}] + ... + [3\alpha a_3 + (n-2)a_2]\big\}\Big]. \end{aligned}$$

(ii) $|z| < \frac{1}{na_0+\alpha a_1}\big[X_5 - (na_0 + \alpha a_1)\big]$ if n is odd, where

$$\begin{aligned} X_5 = \;&-2\Big[\big\{[(n-2)\alpha a_{n-2} + 3a_{n-3}] + ... + [5\alpha a_5 + (n-4)a_4] + [3\alpha a_3 + (n-2)a_2]\big\} \\ &-\big\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\big\}\Big]. \end{aligned}$$

Remark 3 : By taking $\alpha = 0$ in Theorem 2, it reduces to Corollary 3.

Remark 4 : By taking $(i+1)\alpha a_{i+1} + (n-i)a_i > 0$ for $i = 0, 1, 2, ..., n-2$, in Theorem 2, it reduces to Corollary 4.

2. Proof of the Theorems

Proof of Theorem 1 : Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_{m-1}z^{m-1} + a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$ be a polynomial of degree n with real coefficients.

Then the polar derivative of $P(z)$ is

$$\begin{aligned} D_\alpha P(z) \;&=\; [n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} \\ &\quad +... + [4\alpha a_4 + (n-3)a_3]z^3 + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0]. \end{aligned}$$

Now consider the polynomials $J(z)=z^{n-1}D_\alpha P(\frac{1}{z})$ and $R(z)=(z-1)J(z)$ so that

$$\begin{aligned}
R(z) &= (z-1)\Big([n\alpha a_n+a_{n-1}]+[(n-1)\alpha a_{n-1}+2a_{n-2}]z+[(n-2)\alpha a_{n-2}+3a_{n-3}]z^2\\
&\quad+...+[3\alpha a_3+(n-2)a_2]z^{n-3}+[2\alpha a_2+(n-1)a_1]z^{n-2}+[\alpha a_1+na_0]z^{n-1}\Big).\\
R(z) &= [na_0+\alpha a_1]z^n-\Big(\big[na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2\big]z^{n-1}\\
&\quad+\big[(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3\big]z^{n-2}\\
&\quad+...+\big[4a_{n-4}+\{(n-3)\alpha-3\}a_{n-3}-(n-2)\alpha a_{n-2}\big]z^3\\
&\quad+\big[3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}\big]z^2\\
&\quad+\big[2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n\big]z+\big[n\alpha a_n+a_{n-1}\big]\Big)
\end{aligned}$$

Also if $|z|>1$ then $\frac{1}{|z|^{n-i}}<1\ for\ i=0,1,2,...,n-2$. Now

$$\begin{aligned}
|R(z)| &\geq \big|na_0+\alpha a_1\big|\big|z\big|^n-\Big(\big|na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2\big|\big|z\big|^{n-1}\\
&\quad+\big|(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3\big|\big|z\big|^{n-2}\\
&\quad+\big|(n-2)a_2+\{3\alpha-(n-3)\}a_3-4\alpha a_4\big|\big|z\big|^{n-3}\\
&\quad+...+\big|4a_{n-4}+\{(n-3)\alpha-3\}a_{n-3}-(n-2)\alpha a_{n-2}\big|\big|z\big|^3\\
&\quad+\big|3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}\big|z\big|^2\\
&\quad+\big|2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n\big|\big|z\big|+\big|n\alpha a_n+a_{n-1}\big|\Big)\\
&\geq |na_0+\alpha a_1||z|^{n-1}\Bigg[|z|-\frac{1}{|na_0+\alpha a_1|}\big\{|na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2|\\
&\quad+\frac{|(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3|}{|z|}+\frac{|(n-2)a_2+\{3\alpha-(n-3)\}a_3-4\alpha a_4|}{|z|^2}\\
&\quad+...+\frac{|4a_{n-4}+\{(n-3)\alpha-3\}a_{n-3}-(n-2)\alpha a_{n-2}|}{|z|^{n-4}}\\
&\quad+\frac{|3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}|}{|z|^{n-3}}\\
&\quad+\frac{|2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n|}{|z|^{n-2}}+\frac{|n\alpha a_n+a_{n-1}|}{|z|^{n-1}}\}\Bigg]\\
&\geq |na_0+\alpha a_1||z|^{n-1}\Bigg[|z|-\frac{1}{|na_0+\alpha a_1|}\big\{|na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2|\\
&\quad+|(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3|+|(n-2)a_2+\{3\alpha-(n-3)\}a_3-4\alpha a_4|\\
&\quad+...+|4a_{n-4}+\{(n-3)\alpha-3\}a_{n-3}-(n-2)\alpha a_{n-2}|
\end{aligned}$$

$$
\begin{aligned}
& +|3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}| \\
& +|2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n|+|n\alpha a_n+a_{n-1}|\Big\}\Bigg] \\
\geq\ & |na_0+\alpha a_1||z|^{n-1}|z|-\frac{1}{|na_0+\alpha a_1|}\Bigg[\Big\{[2\alpha a_2+\{(n-1)-\alpha\}a_1-na_0] \\
& +[(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3]+[4\alpha a_4+\{(n-3)-3\alpha\}a_3-(n-2)a_2] \\
& +...+[(n-2)\alpha a_{n-2}+\{3-(n-3)\alpha\}a_{n-3}-4a_{n-4}] \\
& +[3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}] \\
& +[n\alpha a_n+\{1-(n-1)\alpha\}a_{n-1}-2a_{n-2}]+|n\alpha a_n+a_{n-1}|\Big\}\Bigg]\ if\ n\ is\ even, \\
=\ & |na_0+\alpha a_1||z|^{n-1}\Bigg[|z|-\frac{1}{|na_0+\alpha a_1|}\big\{|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})+X_1-(na_0+\alpha a_1)\big\}\Bigg] \\
>\ & 0 \\
& if\ |z|>\frac{1}{|na_0+\alpha a_1|}\big[|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})+X_1-(na_0+\alpha a_1)\big],
\end{aligned}
$$

$$
\begin{aligned}
here\ X_1 =\ & 2\big[\big\{[(n-2)\alpha a_{n-2}+3a_{n-3}]+...+[4\alpha a_4+(n-3)a_3]+[2\alpha a_2+(n-1)a_1]\big\} \\
& -\big\{[(n-1)\alpha a_{n-1}+2a_{n-2}]+[(n-3)\alpha a_{n-3}+4a_{n-4}]+...+[3\alpha a_3+(n-2)a_2]\big\}\big]
\end{aligned}
$$

if n is even.

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z|\leq\frac{1}{|na_0+\alpha a_1|}\big[|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})+X_1+(na_0+\alpha a_1)\big].$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of $R(z)$ and hence $J(z)$ lie in

$$|z|\leq\frac{1}{|na_0+\alpha a_1|}\big[|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})+X_1-(na_0+\alpha a_1)\big].$$

Since $D_\alpha P(z)=z^nJ(\frac{1}{z})$ it follows, by replacing z by $\frac{1}{z}$,

then all the zeros of $D_\alpha P(z)$ lie in

$$|z|\geq\frac{1}{|na_0+\alpha a_1|}\big[|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})+X_1-(na_0+\alpha a_1)\big].$$

Hence $D_\alpha P(z)$ does not vanish in the disk

$$|z|<\frac{1}{|na_0+\alpha a_1|}\big[|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})+X_1-(na_0+\alpha a_1)\big],$$

$$\begin{aligned} \textit{where } X_1 &= 2\Big[\ \big\{[(n-2)\alpha a_{n-2} + 3a_{n-3}] + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\big\} \\ &\quad -\big\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + [(n-3)\alpha a_{n-3} + 4a_{n-4}] + ... + [3\alpha a_3 + (n-2)a_2]\big\}\Big] \end{aligned}$$

if n is even.

Similarly we can also prove for odd degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above.

That is if n is odd then $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{|na_0 + \alpha a_1|}\big[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + X_2 + (na_0 + \alpha a_1)\big],$$

$$\begin{aligned} \textit{where} \quad X_2 &= 2\Big[\big\{[(n-2)\alpha a_{n-2} + 3a_{n-3}] + ... + [5\alpha a_5 + (n-4)a_4] + [3\alpha a_3 + (n-2)a_2]\big\} \\ &\quad -\big\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + [(n-3)\alpha a_{n-3} + 4a_{n-4}] + ... + [2\alpha a_2 + (n-1)a_1]\big\}\Big] \end{aligned}$$

if n is odd.

This completes the proof of the Theorem 1.

Proof of Theorem 2 : It is similar to the proof of Theorem 1.

References

[1] Aziz A. and Mohammad Q. G., Zero free regions for polynomials and some generalizations of Eneström-Kakeya theorem, Canad. Math. Bull., 27(3) (1984), 265-272.

[2] Dewan K. K. and Bidkham M., On the Eneström-Kakeya theorem, J. Math. Anal. Appl., 180 (1993), 29-36.

[3] Eneström G., Remarquee sur un théorème relatif aux racines de l'equation $a_n + ... + a_0 = 0$ oü tous les coefficient sont et positifs, Tôhoku, Math. J., 18 (1920), 34-36.

[4] Gulzar M. H., Zero-free regions for polynomials with restricted coefficients, International Journal Of Engineering And Science, 2(6) (2013), 06-10.

[5] Kakeya S., On the limits of the roots of an alegebraic equation with positive coefficient, Tôhoku Math.J, 2, (1912-1913), 140-142.

[6] Ramulu P., Some generalization of Eneström-Kakeya heorem, International Journal of Mathematics and Statistics Invention, 3(2) (2015), 52-59.

[7] Reddy G. L., Ramulu P. and Gangadhar C., Zero-free region for polynomials with restricted coefficients, International Journal of Science and Research, 4(4) (2015), 2937-2942.