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TWO DIMENSIONAL JACOBI-DUNK1 TRANSFORM

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Abstract

In this paper we give the generalization of the analogue [1] of the Jacobi-Dunkl kernel. We have defined Jacobi-Dunkl transform for functions of two variables. We have also derived some analogy of Miyachi's theorem for the Jocobi-Dunkl transform for two variables. Inversion formula is also obtained.

1. Introduction

R. Daher [1] has generalized theorems of Hardy and Miyachi for the Fourier transform on real line to the Jacobi-Dunkl transform. We define the analogue [1] the differential difference operator on R^2 by,

$$\begin{aligned}\wedge_{\alpha,\beta} f(x_1,x_2) &= \frac{\partial}{\partial x_1} f(x_1,x_2) + \frac{\partial}{\partial x_2} f(x_1,x_2) \\ &\quad [(2\alpha+1)\coth(x_1,x_2) + (2\beta+1)\tanh(x_1,x_2)\frac{f(x_1,x_2)-f(-x_1,-x_2)}{2}\end{aligned}$$

where$\alpha \geq \beta \geq 2(-1/2);\ \alpha \neq (1/2)$.

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2. Jacobi-Dunkl Kernel of Two Variables

In this section we define Jacobi-Dunkl kernel of two variables $\psi^{(\alpha,\beta)}_{(\lambda_1,\lambda_2)}$ is the unique C^∞-solution on $\mathbb{R}^2$ of the differential-difference equation given by,

$$\begin{aligned} &\wedge_{\alpha,\beta}(u_1,u_2) = -i\lambda_1\lambda_2 u_1 u_2 \\ &u_1(0,0) = u_2(0,0) = 1 \quad \text{for every} \ \ \lambda_1,\lambda_2 \in \mathbb{C}. \end{aligned} \tag{1}$$

It has also the Laplace integral representation $\mathbb{R}^2$ for every $\lambda_1,\lambda_2 \in \mathbb{C}$ and $\forall\ \ x_1 \times x_2 \in \mathbb{R}^2 \setminus (0,0)$

$$\psi^{(\alpha,\beta)}_{(\lambda_1,\lambda_2)}(x_1,x_2) = \int_{-|x_1\times x_2|}^{|x_1\times x_2|} K(x_1,x_2,x_3,x_4)e^{-i\lambda_1\lambda_2(x_1,x_2,x_3,x_4)}dx_3dx_4 \tag{2}$$

where $K(x_1,x_2)$ the positive is function on $\mathbb{R}^2$ continuous on (x_1,x_2) and satisfies

$$\forall\ \ x_1\times x_2 \in \mathbb{R}^2\setminus(0,0) \int_{\mathbb{R}^2} K(x_1,x_2,x_3,x_4)dx_3dx_4 = 1. \tag{3}$$

By (2) and (3) we deduce that the function $\psi^{(\alpha,\beta)}_{(\lambda_1,\lambda_2)}$ satisfies the following elementary estimate.

Lemma 1 : Assume that $\alpha \geq \beta \geq (-1/2); \alpha \neq (-1/2), \lambda_1,\lambda_2 \in \mathbb{C}$ then

$$\left|\psi^{(\alpha,\beta)}_{(\lambda_1,\lambda_2)}\right| \leq \psi^{(\alpha,\beta)}_{iIm(\lambda_1,\lambda_2)} \tag{4}$$

and

$$\forall\ \ x_1\times x_2\in\mathbb{R}^2, \quad \left|\psi^{(\alpha,\beta)}_{(\lambda_1,\lambda_2)}\right| \leq Me^{|Im(\lambda_1\lambda_2)||x_1x_2|}. \tag{5}$$

Proof : Analogue to [1].

3. Jacobi-Dunkl Transform of Two Variables

In this section we define Jacobi-Dunkl transform of two variables. We denote $\rho = \alpha+\beta+1$ where $\alpha \geq \beta \geq (-/2); \alpha \neq (-1/2)\cdot D(\mathbb{R}^2)$ - the space of all compactly supported C^∞ - functions on $\mathbb{R}^2$ the space of all compactly supported C^∞ - functions g on $\mathbb{R}^2$ which are rapidly decreasing together with their derivatives and $S^r(\mathbb{R}^2)$ $(0 < r \leq 1)$, the generalized Schwartz space[2] defined by $S'(\mathbb{R}^2) = (\cosh(x_1,x_2))^{(-2\rho,r)}S(\mathbb{R}^2)$.

Now we define Jacobi-Dunkl transform of function of two variables $f \in D(\mathbb{R}^2)$ [2] defined by for every $\lambda_1,\lambda_2 \in \mathbb{C}$

$$Ff(\lambda_1,\lambda_2) = \int\int_{R^2} f(x_1,x_2)\psi^{(\alpha,\beta)}_{(\lambda_1,\lambda_2)}\Delta_{\alpha,\beta}(x_1,x_2)dx_1dx_2 \tag{6}$$

where $\Delta_{\alpha,\beta}(x_1, x_2) = (2\sinh(x_1, x_2))^{2\alpha+1}(2\cosh(x_1, x_2))^{2\beta+1}$.

4. Inversion of Jacobi-Dunkl Transform of Two Variables

For $\alpha \geq \beta \geq (-1/2)$; $\alpha \neq (-1/2)$,and $f \in S'(\mathbb{R}^2)$ $(0 < r \leq 1)$, $(x_1, x_2) \in \mathbb{R}^2$. Inversion of Jacobi-Dunkl transform of two variables is given by following formula

$$f(x_1, x_2) = \int_{\mathbb{R}^2} Ff(\lambda_1, \lambda_2)\psi^{(\alpha,\beta)}_{(-\lambda_1,-\lambda_2)}(x_1, x_2)d\sigma(\lambda_1, \lambda_2) \tag{7}$$

where $d\sigma$ is the measure given by

$$d\sigma(\lambda_1, \lambda_2) = \frac{|\lambda_1\lambda_2|}{8\pi(\lambda_1^2\lambda_2^2 - \rho^2)^{(1/2)}|c(\lambda_1^2\lambda_2^2 - \rho^2)|^2}\chi_{\mathbb{R}^2\setminus(-\rho,\rho)}(\lambda_1, \lambda_2 d\lambda_1 d\lambda_2 \tag{8}$$

where $\chi_{\mathbb{R}^2\setminus(-\rho,\rho)}(\lambda_1, \lambda_2)$ is the characteristic function of $\mathbb{R}^2 \setminus (-\rho, \rho)$ and

$$C(\mu) = \frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma((1/2)(\rho+i\mu)\Gamma(1/2)(\alpha-\beta+1+i\mu))}, \quad \mu \in C \setminus \{iN\} \tag{9}$$

is given by [1].

5. The Heat Kernel

Now we define heat kernel E associated with the Jacobi-Dunkl operator for function of two variables.

Definition 3.1 : Let $t > 0$. The heat kernel E_t associated with the Jacobi-Dunkl operator for function of two variables is defined by

$$\forall \;\; x_1, x_2 \in R, \quad E_t(x_1, x_2) = F^{-1}(e^{-t\lambda_1^2\lambda_2^2})(x_1, x_2) \tag{10}$$

this heat kernel E_t has the following properties [1]

(1) for all $t > 0$, E_t is an even positive C^∞ - function on R^2.

(2) $\forall\; t > 0, \;\forall\; \lambda \in R^2, FE_t(\lambda_1, \lambda_2) = e^{-t\lambda_1^2\lambda_2^2}$.

6. Version of the Phragmen-Lindeloff Theorem for Two Variables

For the proof of the main results of next section we require the following two lemma on the two complex variable.

Lemma 6.1 : Let h be an entire function on C such that

$$\forall \;\; z_1, z_2 \in C, \quad |h(z_1,z_2)| \leq Ce^{-|z_1 z_2|^2} \tag{11}$$

and

$$\forall \;\; t \in R, \quad |h(z_1,z_2)| \leq Ce^{at^2} \tag{12}$$

for some positive constants a and c. Then $h(z_1,z_2) = const\ e^{az_1^2 z_2^2},\ z_1, z_2 \in C.$

Proof : Analogues to [1] (See ref.1, Lemma 3.1)

As usual, let us define $\log^+(x) = \log(x)$ if $x > 1$ and $\log^+(x) = 0$ otherwise. We also need the following lemma.

Lemma 6.2 : Suppose g is an entire function and suppose there exists constants $A, B > 0$ such that for all

$$z \in C|g(z_1,z_2)| \leq Ae^{B(Re(z_1)Re(z_2))^2} \tag{13}$$

Also suppose

$$\int_{-\infty}^{\infty} \log^+ |g(z_1,z_2)| < \infty. \tag{14}$$

Then g is a constant function.

Proof : Analogues to [1] (See ref 1 Lemma 5).

7. Main Results

In this section we state and prove analogues of Hardy's Miyachi's theorems for two variables.

7.1. An analog of Hardy's theorem for two variables

Theorem 7.1 : Let f be a measurable function on $\mathbb{R}^2$ such that

$$\forall \;\; x_1, x_2 \in R\ |f(x_1,x_2)| \leq ME_a(x_1,x_2) \tag{15}$$

and

$$\forall \;\; \lambda_1, \lambda_2 \in 1R, \quad |Ff(\lambda_1,\lambda_2)| \leq Me^{-a\lambda_1^2\lambda_2^2} \tag{16}$$

for some constant $a > 0$ and $M > 0$.

Then the function f is a constant multiple of the heat kernel E_a.

Proof : First since by the condition (15), $Ff(\lambda_1, \lambda_2)$ is well defined for all λ_1, λ_2 and Ff is an entire function on C. Moreover using the estimate (4) and (15) for all $\lambda_1, \lambda_2 \in C$ we have

$$\begin{aligned} |Ff(\lambda_1, \lambda_2)| &\leq \int_R |f(x_1, x_2)||\psi^{\alpha,\beta}_{\lambda_1,\lambda_2}(x_1, x_2)|\Delta_{\alpha,\beta}(x_1, x_2)dx_1dx_2 \\ &\leq M\int_R E_a(x_1, x_2)\psi^{\alpha,\beta}_{iIm(\lambda_1\lambda_2)}(x_1, x_2)\Delta_{\alpha,\beta}(x_1, x_2)dx_1dx_2 \\ &= MF(E_a)(iIm(\lambda_1\lambda_2)) \end{aligned}$$

form (10) we have

$$|Ff(\lambda_1, \lambda_2)| \leq Me^{aIm(\lambda_1\lambda_2)^2}. \tag{17}$$

Obtain or $Im(\lambda_1\lambda_2)^2 \leq |\lambda_1\lambda_2|^2$ then we have

$$|Ff(\lambda_1, \lambda_2)|^2 \leq Me^{-a(\lambda_1^2\lambda_2^2)} \text{ for all } \lambda_1, \lambda_2 \in C. \tag{18}$$

We also have by assumption $|Ff(\lambda_1, \lambda_2) \leq Me^{-a(\lambda_1^2\lambda_2^2)}$ for all $\lambda_1, \lambda_2 \in R$ so by Lemma 4 we have $Ff(\lambda_1, \lambda_2) = const.\ e^{-a(\lambda_1^2\lambda^2)2)}$ for $\lambda_1, \lambda_2 \in C$.
Using (10) we prove that $f(x) = const.\ f(x_1, x_2) = E_a(x_1, x_2)$.
Hence the theorem.

7.2 An analogue of Miyachi's theorem for two variables

We denote by

(1) $L^p_{\alpha,\beta}(R^2), p \in (1, \infty)$ the space of measureable function f on R^2 such that

$$\|f\|_{1,\alpha,\beta} = \int_R |f(x_1, x_2)||\psi^{\alpha,\beta}_\lambda(x_1, x_2)|\Delta_{\alpha,\beta}(x_1, x_2)dx_1dx_2 < \infty$$

and

$$\|f\|_{\infty,\alpha,\beta} = ess \sup_{x_1,x_2\in R} |f(x_1, x_2)| < \infty.$$

(2) $L^p(R^2), p \in (1, \infty)$ is defined by in the obvious way.

Now our principal result is as follows:

Theorem 7.3 : Let $a > 0$. Suppose f is a function on R^2 such that

$$E_a^{-1}(x_1, x_2)f(x_1, x_2) \in (L^1_{\alpha,\beta} + L^\infty_{\alpha,\beta})(R^2_x)$$

and

$$\int_{-\infty}^{+\infty} \log^{+}\left(\left|\frac{Ff(\lambda_1,\lambda_2)e^{ea\lambda_1^2\lambda_2^2}}{\xi}\right|\right) d\lambda_1 d\lambda_2 < \infty$$

for some $\xi, 0 < \xi < \infty$. Then f is a constant multiple of the heat kernel E_a.

Proof : We consider

$$Ff(\lambda_1,\lambda_2) = \int\int_{R^2} f(x_1,x_2)\psi^{\alpha,\beta}_{\lambda_1,\lambda_2}(x_1,x_2)\Delta_{\alpha,\beta}(x_1,x_2)dx_1dx_2 \quad \text{for } \lambda_1,\lambda_2 \in C.$$

Using equation (5), we get

$$|Ff(\lambda_1,\lambda_2)| \leq \int\int_{R^2} f(x_1,x_2)\psi^{(\alpha,\beta)}_{im(\lambda_1,\lambda_2)}(x_1x_2)\Delta_{\alpha,\beta}(x_1,x_2)dx_1dx_2.$$

The integrated of the above integral can be written as follows:

$$|E_a^{-1}(x_1,x_2)f(x_1,x_2)|E_a(x_1,x_2)\psi^{(\alpha,\beta)}_{im(\lambda_1,\lambda_2)}(x_1,x_2)\Delta_{|al,\beta}(x_1,x_2).$$

The first factor of which belongs to $(L^1_{\alpha,\beta}+L^\infty_{\alpha,\beta})(R_{x_1,x_2})$, by assumption and the second belongs to $(L^1_{\alpha,\beta}\cap L^\infty_{\alpha,\beta})(R_{x_1,x_2})$.

Hence, $Ff(\lambda_1,\lambda_2)$ is well defined for all $\lambda_1,\lambda_2 \in C$. Moreover,

$$\begin{aligned}
|Ff(\lambda_1,\lambda_2)| &\leq \int_R |E_a^1(x_1,x_2)f(x_1,x_2)|E_a(x_1,x_2)\psi^{\alpha,\beta}_{im(\lambda_1,\lambda_2)}(x_1,x_2)\Delta_{\alpha,\beta}(x_1,x_2)\\
&\leq C\int\int_{R^2} E_a(x_1,x_2)\psi^{\alpha,\beta}_{iIm(\lambda)}(x_1,x_2)\Delta_{\alpha,\beta}(x_1,x_2)dx_1dx_2\\
&\leq CF(E_a)(iIm(\lambda_1\lambda_2))\\
&\leq Ce^{a(I(\lambda_1\lambda_2))^2} \quad \text{with a constant } C\text{, independent of } \lambda_1,,\lambda_2.
\end{aligned}$$

Let $g(\lambda_1,\lambda_2) = e^{\lambda_1^2\lambda_2^2}Ff(\lambda_1\lambda_2)$ this is also an entire function.

By equation (18), we have

$$|g(\lambda_1,\lambda_2)| \leq Ce^{a(Re(\lambda_1\lambda_2))^2}, \quad \text{for} \quad \lambda_1,\lambda_2 \in C.$$

We also have, by assumption,

$$\int_{-\infty}^{+\infty} \log^{+}\left(\frac{|g(\lambda_1,\lambda_2)|}{\xi}\right) d\lambda_1 d\lambda_2 < \infty.$$

Hence, applying the crucial Lemma 5 to the function $g(\lambda_1,\lambda_2)/\xi$ we see that $g(\lambda_1,\lambda_2) =$ constant $= K$ or equivalently

$$Ff(\lambda_1,\lambda_2) = Ke^{-a\lambda_1^2\lambda_2^2}.$$

From (10), by Fourier inversion we get that the function f satisfies

$$f(x_1, x_2) = KE_a(x_1, x_2).$$

Hence theorem is proved.

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